REAL ALGEBRAIC MORPHISMS ON 2-DIMENSIONAL CONIC BUNDLES

FRÉDÉRIC MANGOLTE

Abstract. Given two nonsingular real algebraic varieties V and W, we consider the problem of deciding whether a smooth map $f : V \to W$ can be approximated by regular maps in the space of $\mathcal{C}^\infty$ mappings from V to W in the $\mathcal{C}^\infty$ topology.

Our main result is a complete solution to this problem in case W is the usual 2-dimensional sphere and V is a real algebraic surface of negative Kodaira dimension.

1. Introduction

We deal with regular maps between real algebraic varieties. Here the term *real algebraic variety* stands for a locally ringed space isomorphic to a locally closed subset of $\mathbb{P}^n(\mathbb{R})$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions. Morphisms between real algebraic varieties are called *regular maps*.

An equivalent description of real algebraic varieties can be obtained using quasi-projective varieties defined over $\mathbb{R}$. Given such a variety X, the Galois group $G = \{1, \sigma\}$ of $\mathbb{C}|\mathbb{R}$ acts on $X(\mathbb{C})$, the set of complex points of X, via an antiholomorphic involution. The real part $X(\mathbb{R})$ is then precisely the set of fixed points under this action. If $X(\mathbb{R})$ is Zariski dense in X, then we consider it as a real algebraic variety whose structure sheaf is the restriction of the structure sheaf of X. Therefore, a regular map $V \to W$ in the above sense is the restriction of a rational map $X(\mathbb{C}) \dashrightarrow Y(\mathbb{C})$ with no poles on $V = X(\mathbb{R})$.

Every real algebraic variety is isomorphic to a Zariski closed subvariety of $\mathbb{R}^n$. Thus, all topological notions about real algebraic varieties will refer to the Euclidean topology of $\mathbb{R}^n$.

Given two nonsingular real algebraic varieties V and W, with V compact, we consider the set $\mathcal{R}(V, W)$ of all regular maps from V into W as a subset of the space $\mathcal{C}^\infty(V, W)$ of all $\mathcal{C}^\infty$ maps from V into W equipped with the $\mathcal{C}^\infty$ topology. We want to study which $\mathcal{C}^\infty$ maps from V into W can be approximated by regular maps. The classical Stone-Weierstrass approximation theorem implies that all $\mathcal{C}^\infty$ maps from V into W can be approximated by regular maps when $W = \mathbb{R}^m$ for some m. In this paper, we will mainly consider the case when $W = S^2$ is the usual two dimensional Euclidean sphere.

The main achievement of this paper is a complete answer to the approximation problem of smooth maps from a real algebraic surface of negative Kodaira dimension into the 2-sphere.

In dimension less than three, an algebraic variety of negative Kodaira dimension is $\mathbb{C}$-uniruled. In dimension two, such a variety is $\mathbb{C}$-ruled. By definition, a $\mathbb{C}$-ruled surface X is $\mathbb{C}$-birationally equivalent to a product $\mathbb{P}^1 \times B$, where B is a complex

1991 *Mathematics Subject Classification.* 14P05 14P25 14J26.

I would like to thank J. Bochnak for very helpful discussions. Many thanks also to C. Raffalli and P. Verovic for their careful reading.

algebraic curve. When the genus $g(B)$ of B is non vanishing, the surface X admits a relatively minimal model Y over $\mathbb{C}$ endowed with a $\mathbb{P}^1$-bundle structure $Y \longrightarrow B$.

When X is defined over $\mathbb{R}$, X may not be $\mathbb{R}$-birational to a product though it is $\mathbb{C}$-birational to $\mathbb{P}^1 \times B$; consider for example a maximal real Del Pezzo surface of degree 2 or 1.

Even when $g(B) \neq 0$, it may occur that no relatively $\mathbb{R}$-minimal model Y of X is a $\mathbb{P}^1$-bundle. In this case, the surface X admits only relatively $\mathbb{R}$-minimal models which are not $\mathbb{C}$-minimal: these are the real conic bundles, see Section 4.

Among all the surfaces of negative Kodaira dimension are the rational surfaces. There are several ways to define real algebraic varieties, hence several ways to define real rational surfaces. A real algebraic surface V is $\mathbb{C}$-*rational* (or geometrically rational) if its complexification X is $\mathbb{C}$-birationally equivalent to $\mathbb{P}^2_{\mathbb{C}}$. Similarly, the surface V is $\mathbb{R}$-*rational* if V is $\mathbb{R}$-birationally equivalent to $\mathbb{P}^2_{\mathbb{R}}$.

The approximation problem of smooth maps from $\mathbb{R}$-rational surfaces into the 2-sphere was solved by J. Bochnak and W. Kucharz [BK87a, BK87b].

In collaboration with N. Joglar, we generalized this to the case when the source space is a $\mathbb{C}$-rational real algebraic surface [JM03].

In this paper, we solve the problem for all the surfaces of negative Kodaira dimension.

Let X be a real algebraic surface of negative Kodaira dimension which is not $\mathbb{C}$-rational. Hence X admits a real ruling $\rho\colon X \longrightarrow B$. Recall that a connected component of $X(\mathbb{R})$ may be diffeomorphic to a torus, a sphere or any nonorientable surface. We denote by K' the (possibly empty) set of connected components of $X(\mathbb{R})$ which are diffeomorphic to the Klein bottle and whose image by ρ is a connected component of $B(\mathbb{R})$.

Theorem 1.1. *Let X be a $\mathbb{C}$-ruled non $\mathbb{C}$-rational real algebraic surface. Given a smooth map $f\colon X(\mathbb{R}) \longrightarrow S^2$, the following conditions are equivalent:*

(1) *f can be approximated by regular maps;*
(2) *f is homotopic to a regular map;*
(3) *for each component M of $X(\mathbb{R})$ diffeomorphic to a torus, $\deg(f)_{|M} = 0$ and for each pair of components belonging to K', $\deg_{\mathbb{Z}/2}(f)_{|M} = \deg_{\mathbb{Z}/2}(f)_{|N}$.*

In [Ku99], W. Kucharz gave another kind of generalization of his result with J. Bochnak about $\mathbb{R}$-rational surfaces. Namely, he solved the approximation problem of smooth maps from $\mathbb{R}$-rational surfaces into $\mathbb{R}$-rational surfaces.

We extend this result as follows (recall that a $\mathbb{C}$-rational real algebraic surface X is $\mathbb{R}$-rational if and only if $X(\mathbb{R})$ is connected):

Theorem 1.2. *Let $V = X(\mathbb{R})$ and $W = Y(\mathbb{R})$ be connected real algebraic surfaces such that X is $\mathbb{C}$-ruled and Y is $\mathbb{C}$-rational. Then the space $\mathcal{R}(V, W)$ is dense in the space $\mathcal{C}^\infty(V, W)$, except when V is diffeomorphic to a torus and W is diffeomorphic to a sphere.*

In the latter case, the closure of $\mathcal{R}(V, W)$ in $\mathcal{C}^\infty(V, W)$ consists precisely of the null homotopic maps.

Furthermore, we have answered a question raised by J. Bochnak. The following result is largely independent of the previous ones.

Theorem 1.3. *Let V be a $\mathbb{R}$-rational real algebraic surface diffeomorphic to the Klein bottle. Then V is biregularly isomorphic to the blow-up of the real projective plane over one point. In other words, a $\mathbb{R}$-rational Klein surface always admits a non minimal smooth complexification.*

This result fits with the more general setting: a rational model of a connected compact variety M is a $\mathbb{R}$-rational real algebraic surface diffeomorphic to M.

By Comessatti classification, if M is orientable its genus must be less than 2. It is known that the sphere and the torus each admits a unique rational model modulo biregular isomorphism. Thanks to the latter theorem, there is also only one rational model for the Klein bottle. Hence, the next natural question is: "is there a genus h for which the nonorientable surface of Euler characteristic $1-h$ admits several rational models?".

One of the main tools used in the proof of Theorems 1.1 and 1.2 is a new characterization of a classical invariant of real algebraic varieties used in the approximation problem in case the target space is the usual sphere.

Given a compact nonsingular real algebraic variety V, consider a smooth projective variety X over $\mathbb{R}$, such that V and $X(\mathbb{R})$ are isomorphic as real algebraic varieties. We denote by $H^2_{\mathrm{alg}}(X(\mathbb{C}),\mathbb{Z})$ the subgroup of $H^2(X(\mathbb{C}),\mathbb{Z})$ that consists of the cohomology classes that are Poincaré dual to the homology classes in $H_{2n-2}(X(\mathbb{C}),\mathbb{Z})$ represented by divisors in $X_{\mathbb{C}}$. We set

$$H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z}) = i^*(H^2_{\mathrm{alg}}(X(\mathbb{C}),\mathbb{Z}))\ ,$$

where $i : X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$ is the inclusion map. We will denote by $\Gamma(X)$ the quotient group $H^2(X(\mathbb{R}),\mathbb{Z})/H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$. It is easy to check that for a given nonsingular real algebraic variety V, the group $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$ does not depend on the associated variety X. We can identify V and $X(\mathbb{R})$ and set

$$H^2_{\mathbb{C}-\mathrm{alg}}(V,\mathbb{Z}) = H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$$

We will use the following notation:

$$\Gamma(V) = \Gamma(X)\ .$$

There is a close connection between the subgroup $H^2_{\mathbb{C}-\mathrm{alg}}(V,\mathbb{Z})$ and the topological closure of the space $\mathcal{R}(V,S^2)$ in $\mathcal{C}^\infty(V,S^2)$. More precisely, the following result is well-known, cf. [BCR98, Chapter 13] and [BBK89].

Proposition 1.4. *Let V be a compact nonsingular real algebraic variety. A given $\mathcal{C}^\infty$ map $f\colon V \to S^2$ can be approximated by regular maps in the $\mathcal{C}^\infty$ topology, if and only if $f^*(\kappa) \in H^2_{\mathbb{C}-\mathrm{alg}}(V,\mathbb{Z})$. Here κ is a fixed generator of the group $H^2(S^2,\mathbb{Z})$.*

In terms of the quotient Γ, here are the already known cases:

Theorem 1.5 ([BK87a, BK87b]). *Let V be a $\mathbb{R}$-rational real algebraic surface. Then*

$$\Gamma(V) = \begin{cases} \mathbb{Z} & \text{if } V \text{ is diffeomorphic to } S^1\times S^1 \\ 0 & \text{in all other cases.} \end{cases}$$

On the other hand, a smooth projective surface X is a *Del Pezzo surface* iff X is irreducible and the anticanonical divisor $-K_X$ is ample. The degree of X is $\deg(X) = K_X^2$. For Del Pezzo surfaces, it is known that $1 \le \deg(X) \le 9$. A real Del Pezzo surface X is $\mathbb{C}$-rational and is not $\mathbb{R}$-rational when $X(\mathbb{R})$ is not connected.

Theorem 1.6 ([JM03]). *Let V be a real algebraic surface biregularly isomorphic to the real part of a maximal real Del Pezzo surface of degree 2, then*

$$\Gamma(V) = \mathbb{Z}/2\ .$$

Theorem 1.7 ([JM03]). *Let $V = X(\mathbb{R})$ be a $\mathbb{C}$-rational real algebraic surface. Then*

$$\Gamma(V) = \begin{cases} \mathbb{Z} & \text{if } V \text{ is diffeomorphic to } S^1 \times S^1 \\ \mathbb{Z}/2 & \text{if } V \text{ is as in Theorem 1.6} \\ 0 & \text{in all other cases.} \end{cases}$$

Convention. A real algebraic variety is smooth projective and geometrically irreducible, unless otherwise stated.

2. Algebraic morphisms to the standard sphere

Let again $i : X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$ be the canonical injection of the set of real points into the set of complex points of a real algebraic surface. Consider the induced restriction morphism

$$i^* \colon H^2(X(\mathbb{C}), \mathbb{Z}) \to H^2(X(\mathbb{R}), \mathbb{Z}) \ .$$

We will use the notation X^{or} for the disjoint union of the orientable connected components of the real part and X^{nor} for the nonorientable part. The morphism i^* has the natural splitting $H^2(X(\mathbb{C}), \mathbb{Z}) \to H^2(X^{or}, \mathbb{Z}) \oplus H^2(X^{nor}, \mathbb{Z})$. Since X^{nor} is nonorientable of dimension 2, $H^2(X^{nor}, \mathbb{Z})$ is canonically isomorphic to the group $H^2(X^{nor}, \mathbb{Z}/2)$ by reduction modulo 2. To see this, apply twice the universal-coefficient theorem [Sp66, 5.5.10].

We will identify the group $H^2(X(\mathbb{R}), \mathbb{Z})$ with the direct sum $H^2(X^{or}, \mathbb{Z}) \oplus H^2(X^{nor}, \mathbb{Z}/2)$ and still use the notation i^* for the composed morphism

$$H^2(X(\mathbb{C}), \mathbb{Z}) \to H^2(X^{or}, \mathbb{Z}) \oplus H^2(X^{nor}, \mathbb{Z}/2) \ .$$

The manifolds $X(\mathbb{C})$ and $X(\mathbb{R})$ are compact and $X(\mathbb{C})$ is orientable. The Gysin morphism $i_!$ can be defined by the commutative diagram:

$$\begin{tikzcd} H^2(X(\mathbb{C}), \mathbb{Z}) \arrow[r, "i^*"] \arrow[d, "D_{\mathbb{C}}"', "\cong"] & H^2(X^{or}, \mathbb{Z}) \oplus H^2(X^{nor}, \mathbb{Z}/2) \arrow[d, "D_{\mathbb{R}}^{or} \oplus D_{\mathbb{R}}^{nor}"', "\cong"] \\ H_2(X(\mathbb{C}), \mathbb{Z}) \arrow[r, "i_!"] & H_0(X^{or}, \mathbb{Z}) \oplus H_0(X^{nor}, \mathbb{Z}/2) \ , \end{tikzcd} \tag{2.1}$$

where the isomorphisms $D_{\mathbb{C}}$, $D_{\mathbb{R}}^{or}$ and $D_{\mathbb{R}}^{nor}$ come from Poincaré duality applied to the orientable 4-dimensional manifold $X(\mathbb{C})$, the orientable 2-dimensional manifold X^{or} and the nonorientable 2-dimensional manifold X^{nor}.

Let S and M be two transverse oriented submanifolds in an oriented manifold X. We attach $+1$ to a point $P \in S \cap M$ if the orientation of the tangent space $T_P X$ coincide with the orientation given by the direct oriented sum $T_P S \oplus T_P M$ and -1 otherwise. With this convention in mind, we obtain a well-defined class $[S \pitchfork M]$ in $H_0(M, \mathbb{Z})$. Now if M is nonorientable, the class $[S \pitchfork M]$ is well-defined modulo 2 in $H_0(M, \mathbb{Z}/2)$ [Hr76]. The following lemma is an exercise in algebraic topology.

Lemma 2.2. *Let S be an oriented 2-dimensional closed submanifold of $X(\mathbb{C})$ transverse to $X(\mathbb{R})$, denote by $[S]$ its fundamental class in $H_2(X(\mathbb{C}), \mathbb{Z})$, then*

$$i_!([S]) = [S \pitchfork X^{or}] \oplus [S \pitchfork X^{nor}] \ .$$

Let X be real algebraic surface and suppose that $X(\mathbb{R}) \neq \varnothing$. Denote by $\{M_j\}_{j \in J}$ the set of connected components of $X(\mathbb{R})$. The $\mathbb{Z}$-module $H^2(X(\mathbb{R}), \mathbb{Z})$ splits into a direct sum $\oplus_{j \in J} H^2(M_j, \mathbb{Z})$.

Let $J' \subset J$ be a subset and l be a class in $H^2(X(\mathbb{C}), \mathbb{Z})$, we will call the image of l by the composed map $H^2(X(\mathbb{C}), \mathbb{Z}) \to H^2(X(\mathbb{R}), \mathbb{Z}) \to H^2(\oplus_{j \in J'} M_j, \mathbb{Z})$ the restriction of $i^*(l)$ to $\oplus_{j \in J'} M_j$. For a connected component M_j of $X(\mathbb{R})$, we will say that a generator class η_j of $H^2(M_j,\mathbb{Z})$ belongs to $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$ iff the class $1\eta_j \oplus \bigoplus_{a\neq j} 0\eta_a$ belongs to $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$.

3. Birational equivalence, orientability and regular maps

Let X be a smooth projective surface over $\mathbb{R}$. We say that a smooth complex curve E of $X_{\mathbb{C}}$ is a (-1)-*curve* if it is rational and $E^2=-1$. If a (-1)-curve E is defined over $\mathbb{R}$, there exists a blowdown $\pi: X\to Y$ over $\mathbb{R}$ onto a smooth surface such that E contracts to a real point $P=\pi(E)\in Y(\mathbb{R})$ (in particular, $Y(\mathbb{R})$ and $X(\mathbb{R})$ are not empty and have the same number of connected components). If E is not defined over $\mathbb{R}$, then $\sigma(E)$ is another (-1)-curve. If the intersection number $E\cdot\sigma(E)=0$, then we can blowdown over $\mathbb{R}$ the divisor $E+\sigma(E)$. We say that a smooth projective surface over $\mathbb{R}$ is relatively $\mathbb{R}$-*minimal* if it contains neither real (-1)-curves nor pairs of disjoint complex conjugated (-1)-curves. If $E\cdot\sigma(E)\neq 0$, then we cannot blowdown $E+\sigma(E)$ over $\mathbb{R}$ and the surface can be $\mathbb{R}$-minimal but not $\mathbb{C}$-minimal, see the next section.

Let L be a *real* algebraic curve on an algebraic surface X. There are two algebraic bundles naturally associated to L. Namely the $\mathbb{C}$-line bundle $\mathcal{E}=\mathcal{O}_X(L)$ over $X(\mathbb{C})$ and the $\mathbb{R}$-line bundle $\mathcal{L}$ over $X(\mathbb{R})$ satisfying the relation

$$\mathcal{L}\otimes\mathbb{C}=\mathcal{E}_{|Y(\mathbb{R})}\ . \tag{3.1}$$

We will use the first Chern class $c_1(L)=c_1(\mathcal{O}_X(L))$ in $H^2(X(\mathbb{C}),\mathbb{Z})$ and the first Stiefel-Whitney class $w_1(L(\mathbb{R}))=w_1(\mathcal{L})$ in $H^1(X(\mathbb{R}),\mathbb{Z}/2)$. We denote by $\beta\colon H^1(X(\mathbb{R}),\mathbb{Z}/2)\to H^2(X(\mathbb{R}),\mathbb{Z})$ the *Bockstein* homomorphism induced in cohomology by the usual exact sequence

$$0\to\mathbb{Z}\xrightarrow{\times 2}\mathbb{Z}\to\mathbb{Z}/2\to 0\ .$$

Lemma 3.2. *Let X be a real algebraic surface and let L be a real algebraic curve on X, then $i^*(c_1(L))=\beta\circ w_1(L(\mathbb{R}))$ in $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$. In particular, the class $i^*(c_1(L))$ is 2-torsion.*

Proof. From (3.1), we get $i^*(c_1(\mathcal{E}))=c_1(\mathcal{L}\otimes\mathbb{C})$ by functoriality of Chern classes. Since $\mathcal{L}\oplus\mathcal{L}$ and $\mathcal{L}\otimes\mathbb{C}$ are naturally isomorphic as oriented real vector bundles and $c_1(\mathcal{L}\otimes\mathbb{C})=\beta\circ w_1(\mathcal{L})$ [MS74, Problem 15.D and Lemma 14.9], the lemma follows. ☐

Corollary 3.3. *For a real curve L, the class $i^*(c_1(L))\neq 0$ in $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$ if and only if there exists a connected component M of $X(\mathbb{R})$ such that $\deg_M(w_1(\mathcal{L})^2)$ is odd. In particular, if $L\cdot L$ is odd, $i^*(c_1(L))$ is a nontrivial class of order 2 in $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$.*

Proof. The image of the Bokstein homomorphism is given by

$$\beta(w_1(\mathcal{L}))=w_1(\mathcal{L})\cup w_1(\mathcal{L})$$

thanks to the Whitney duality theorem. Furthermore, we have $\deg(w_1(\mathcal{L})^2)\equiv L\cdot L \mod 2$. Indeed $\deg(w_1(L(\mathbb{R}))^2)\equiv L(\mathbb{R})\cdot L(\mathbb{R}) \mod 2$ in $H^1(X(\mathbb{R}),\mathbb{Z}/2)$ and $L\cdot L\equiv L(\mathbb{R})\cdot L(\mathbb{R}) \mod 2$ [Si89, Chap. III]. ☐

Remark 3.4. This result was known in case $X(\mathbb{R})$ is connected [BCR98, Th. 12.6.13].

Proposition 3.5. *Let X be a real algebraic surface containing a (-1)-curve L defined over $\mathbb{R}$. There is only one connected component M of $X(\mathbb{R})$ meeting $L(\mathbb{R})$. Furthermore M must be nonorientable and we have*

$$\eta_M=i^*(c_1(L))\ .$$

Hence, the generator class η_M of $H^2(M,\mathbb{Z})$ is a nontrivial 2-torsion class of $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$.

Proof. By Corollary 3.3, $i^*(c_1(L))$ is a nontrivial 2-torsion class. Another consequence of $L\cdot L=-1$ is that L must have a real point. Let M be a connected component of $X(\mathbb{R})$ having a nontrivial intersection with $L(\mathbb{R})$. The curve L is smooth and rational, hence have a connected real part. Then $L(\mathbb{R})\subset M$ and M must be nonorientable and finally $\eta_M=i^*(c_1(L))$. □

We recovered the well-known fact that Γ is not a birational invariant. But we will prove more: Γ is not even an invariant of relative minimal models, see Theorem 4.14 and Corollary 4.15.

Given a dominant $\mathbb{R}$-birational morphism $X\longrightarrow Y$ between smooth real algebraic surfaces, we have a natural injection $\Gamma(X)\hookrightarrow\Gamma(Y)$. In case X admits a unique minimal model, i.e. when $\mathrm{kod}(X)\geq 0$, we can reduce the computation of $\Gamma(X)$ to that of $\Gamma(Y)$.

In case X is $\mathbb{C}$-ruled, the result depends on the real minimal model and the dominant map.

Proposition 3.6. *Let X be a real algebraic surface and denote by $\mathcal{K}_X$ its canonical line bundle. The class $i^*(c_1(\mathcal{K}_X))$ is zero in $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$ if and only if the Euler characteristic $\chi(M)$ is even for any nonorientable component M.*

Proof. Indeed, if $X(\mathbb{R})\neq\varnothing$, $\mathcal{K}_X$ is representable by a real divisor and $i^*(c_1(\mathcal{K}_X))$ belongs to $H^2(X^{nor},\mathbb{Z})$ by Lemma 3.2. We have then

$$i^*(c_1(\mathcal{K}_X))\equiv_{\mathrm{mod}\ 2} w_2(TX(\mathbb{R}))$$

and we conclude by [MS74, Cor. 11.12] about Stiefel-Whitney numbers. □

4. Real conic bundles over curves

Let X be a smooth real algebraic surface and B a smooth real algebraic curve. A connected component M of $X(\mathbb{R})$ is said to be a *spherical* (resp. *torus*, resp. *Klein*) component if M is diffeomorphic to the sphere S^2 (resp. the torus, resp. the Klein bottle).

Definition 4.1. *A morphism $\rho\colon X\to B$ is a* ruling *iff the generic fiber is isomorphic to $\mathbb{P}^1$. The morphism ρ is a* conic bundle *iff every fiber is isomorphic to a plane conic.*

When the map ρ is defined over $\mathbb{R}$, we will say that a fiber of ρ is *real* if it is located over $B(\mathbb{R})$ and *imaginary* otherwise.

A ruling is $\mathbb{C}$-*minimal* iff no fiber contains a (-1)-curve. A real ruling is $\mathbb{R}$-*minimal* iff no real fiber contains a *real* (-1)-curve and no imaginary fiber contains a (-1)-curve. A $\mathbb{C}$-minimal real ruling is clearly $\mathbb{R}$-minimal but the converse does not hold in general. A $\mathbb{C}$-minimal ruling is isomorphic to a locally trivial $\mathbb{P}^1$-bundle. A $\mathbb{R}$-minimal ruling is a real conic bundle.

Recalling that $\mathbb{K}$-ruled means $\mathbb{K}$-birational to a product $\mathbb{P}^1\times B$, we have:

Proposition 4.2. *A given $\mathbb{R}$-minimal, $\mathbb{C}$-ruled and non $\mathbb{C}$-rational real algebraic surface is $\mathbb{R}$-ruled if and only if it is $\mathbb{C}$-minimal.*

A surface X endowed with a minimal ruling (over $\mathbb{C}$ or over $\mathbb{R}$) in the above sense is not a minimal model in the sense of Mori theory, it is only relatively minimal when $g(B)\neq 0$. Indeed, there exist many birationally equivalent minimal rulings. A birational equivalence between two minimal ruling is a composition of elementary transformations. Over $\mathbb{C}$, an elementary transformation centered at a point p is the blow-up centered at p composed by the contraction of the strict transform of the fiber containing p.

Over $\mathbb{R}$, there are two kinds of elementary transformations. We denote by elm_p the elementary transformation centered at a *real* point p that is smooth in $X_{\rho(p)}$ and by $\mathrm{elm}_{p,\sigma(p)}$ the composition of the elementary transformations centered at p and $\sigma(p)$ provided that $X_{\rho(p)}$ and $X_{\rho(\sigma(p))}$ are distinct conjugated fibres.

We will use the following classification theorem:

Theorem 4.3. *Let X be a smooth real surface with a real ruling $\rho\colon X \to B$ over a smooth real curve B. Then X is $\mathbb{R}$-birational to the smooth $\mathbb{R}$-minimal projective completion X^g of the real conic bundle defined in some affine open subset of $\mathbb{A}^2 \times B$ by an equation*

$$x^2 + y^2 = g(z) \ , \tag{4.4}$$

where g is a real rational function over B with no pole in $B(\mathbb{R})$, and whose all real zeros are simple.

Proof. See [Si89, V.2] and [Si89, VI.3]. □

Remark 4.5. Note the number of real zeros of g that belong to a connected component B_1 of $B(\mathbb{R})$ is even. Indeed, the function g changes sign in the neighborhood of a zero in the topological circle B_1.

Proposition 4.6. *Denote by n the number of connected components of $B(\mathbb{R})$ and by $2s$ the number of real zeros of g. Then the real part $X^g(\mathbb{R})$ of X^g is diffeomorphic to the disjoint union of t tori and s spheres, where t satisfies $t \leq n$.*

Proof. From Equation 4.4, the topology of $X^g(\mathbb{R})$ is easy to understand. The real zeros $\{z_l\}_{1\leq l\leq 2s}$ of the function g determine s connected arcs in $B(\mathbb{R})$ over which the real fibers $X^g_z(\mathbb{R})$ are not empty. Over each of these arcs, there is a connected component of $X^g(\mathbb{R})$ which is homeomorphic to a sphere. The torus components are located over components of $B(\mathbb{R})$ where g is strictly positive. □

Let M be a spherical component of $X^g(\mathbb{R})$ and $X^g_{z_l}$ be a real fiber over a zero of g such that $X^g_{z_l}(\mathbb{R}) \subset M$. The real singular fiber $X^g_{z_l}$ is the union of two complex conjugated (-1)-curves E and $\sigma(E)$ whose intersection point p is the only real point of the fiber.

Lemma 4.7. *Let X be a $\mathbb{C}$-ruled surface defined over $\mathbb{R}$, then*

$$H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z}) = \mathrm{Im}\, i^* \ .$$

Proof. By definition, we have $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z}) = i^*(\mathrm{NS}(X_{\mathbb{C}}))$. Moreover, for a complex nonsingular variety V, we have the long exact sequence coming from the exponential exact sequence. In addition, considering the isomorphism $\mathrm{Pic}(V) \cong H^1(V,\mathcal{O}^*)$, we obtain the following exact sequence

$$\cdots \to H^1(V,\mathcal{O}) \to \mathrm{Pic}(V) \xrightarrow{c_1} H^2(V,\mathbb{Z}) \to H^2(V,\mathcal{O}) \to \cdots$$

The Lemma is now clear since $\mathrm{NS}(V) = c_1(\mathrm{Pic}(V))$ and, for a $\mathbb{C}$-ruled surface V, $\dim H^2(V,\mathcal{O}) = 0$. □

Let X be a $\mathbb{R}$-minimal conic bundle, by Theorem 4.3, there exist a surface X^g and a finite sequence T of real elementary transformations such that $T(X^g) = X$.

The connected components $\{M_j\}_{j\in J}$ of $X(\mathbb{R})$ are then spherical, toral or of Klein type. We call respectively $S \subset J, T \subset J, K \subset J$ the subsets of indexes corresponding to the spherical, torus and Klein components respectively. In particular, we have $X^{nor} = \oplus_{j\in K} M_j$.

Now, for each spherical component M_j of $X(\mathbb{R})$, there exist two singular fibers $E_j^c+\sigma(E_j^c)$, $c=1,2$, such that $E_j^c\cap\sigma(E_j^c)\in M_j$ is reduced to a single real point. Let us denote by $\mathcal{E}_j^c$ the associated algebraic $\mathbb{C}$-line bundles and by N the submodule of $H^2(X(\mathbb{C}),\mathbb{Z})$ generated by the $2s$ classes $\{c_1(\mathcal{E}_j^c)\}_{c\in\{1,2\},j\in S}$.

Lemma 4.8. *Let X be a $\mathbb{R}$-minimal conic bundle.*

(1) *The Néron-Severi group $\mathrm{NS}(X_\mathbb{C})$ is generated by N, the class f of a fiber and the class h of a section.*
(2) *The canonical class is given by*
$$c_1(\mathcal{K}_X)=rf-2h+\sum_{c\in\{1,2\},j\in S}c_1(\mathcal{E}_j^c)$$
for some integer $r\in\mathbb{Z}$.

Proof. Indeed, over $\mathbb{C}$, we can blow down the curves E_j^c to obtain a $\mathbb{C}$-minimal complex ruled surface $\rho'\colon Y\longrightarrow B$ whose Néron-Severi group is generated by the class f' of a fiber and the class h' of a section. Moreover, the canonical class $c_1(\mathcal{K}_Y)$ is a linear combination
$$rf'-2h' \tag{4.9}$$
for some integer $r\in\mathbb{Z}$ [B78, III.18].

The strict transform of a generic fiber of Y is a fiber of X and the strict transform of a section of Y is a section of X. Furthermore, for a blow-up centered at $p\in Y$, the total transform of the fiber $Y_{\rho'(p)}$ is a singular fiber for $X\to B$ of the form $E+\sigma(E)$, where E is a (-1)-curve.

Hence, the group $\mathrm{NS}(X_\mathbb{C})$ is generated by the classes $c_1(E_j^c)$, the class f of a fiber and the class h of a section. Furthermore, we deduce from (4.9) that $c_1(\mathcal{K}_X)=rf-2h+\sum c_1(\mathcal{E}_j^c)$. ☐

Lemma 4.10. *Given any $\mathbb{R}$-minimal conic bundle $X\to B$, the canonical class and the class of a fiber satisfy $i^*(\mathcal{K}_X)=0$ and $i^*(f)=0$ in $H^2(X(\mathbb{R}),\mathbb{Z})$.*

Proof. We may assume that $X(\mathbb{R})\neq\varnothing$ hence there exist a fiber F of ρ and a canonical divisor which are real. Then $F.F=0$ as a fiber and the conclusion about f follows from Corollary 3.3. By Lemma 3.2, $i^*(\mathcal{K}_X)$ is a 2-torsion class, hence trivial when restricted to an orientable component. Furthermore, the restriction of $i^*(\mathcal{K}_X)$ to a Klein component $M_j, j\in K$, is trivial as $w_2(M_j)=0$. ☐

Lemma 4.11. *For each spherical component $M_j\subset X(\mathbb{R})$, any generator class η_j of $H^2(M_j,\mathbb{Z})$ belongs to $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$. More precisely, we have*
$$i^*(N)=\oplus_{j\in S}H^2(M_j,\mathbb{Z})\ .$$

Proof. By Theorem 4.3, there exist a real ruling $\rho'\colon X^g\longrightarrow B$ and a finite sequence of real elementary transformations $T(X^g)=X$, where X^g is the $\mathbb{R}$-minimal projective completion of the conic bundle defined by
$$\{(x,y,z)\in\mathbb{A}^2\times B\,|\,x^2+y^2=g(z)\}\quad\text{and}\quad\rho'(x,y,z)=z\ .$$

As ρ and ρ' are $\mathbb{R}$-minimal, we may assume that there is no center of elementary transformation of T that belongs to a reducible fiber. In particular, restricted to a neighborhood of a spherical component M of $X^g(\mathbb{R})$ in $X^g(\mathbb{C})$, T is a real isomorphism.

Let X^g_z be a real fiber over a zero of g such that $X^g_z(\mathbb{R})\subset M$. The real singular fiber X^g_z is the union of two complex conjugated lines E and $\sigma(E)$ whose intersection point p is the only real point of the fiber. The tangent plane to $X^g(\mathbb{R})$ at p is generated by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial y_1}$, where $x=x_1+ix_2$ and $y=y_1+iy_2$. It is easy to check that the tangent plane to E at p is generated by $i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}$. Then E is transverse to $X^g(\mathbb{R})$ at p in $X^g(\mathbb{C})$.

Hence $T(E)$ is transverse to $X(\mathbb{R})$ at $T(p)$ in $X(\mathbb{C})$. By Lemma 2.2, the image of $[T(E)]$ by the Gysin morphism $i_!$ is a generator of

$$H_0(T(M),\mathbb{Z}) \hookrightarrow H_0(X^{or},\mathbb{Z}) \oplus H_0(X^{nor},\mathbb{Z}/2)\ .$$

Therefore, we conclude by using the commutative diagram 2.1. $\square$

Lemma 4.12. *Let M_j be a torus component. Then for any generator class η_j of $H^2(M_j,\mathbb{Z})$ we have $\eta_j \notin H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$.*

Proof. By Lemmas 4.8, 4.10 and 4.11, it suffices to prove that the restriction of $i^*(h)$ to a torus component is trivial.

The canonical class $c_1(\mathcal{K}_X)$ is a linear combination $rf - 2h + \sum c_1(\mathcal{E}_j^c)$ for some integer $r \in \mathbb{Z}$. From Lemma 4.11 and Lemma 4.10, the restrictions of $i^*(2h)$ and $i^*(\mathcal{K}_X)$ to a torus component are equal. Moreover, the class $i^*(\mathcal{K}_X)$ is trivial when restricted to an orientable component. The restriction of $i^*(h)$ to a torus component is then a 2-torsion class, hence trivial. $\square$

Lemma 4.13. *Let h be the class of a section on a $\mathbb{R}$-minimal conic bundle X, then the restriction of $i^*(h)$ to the nonorientable part is the class $\oplus_{j\in K}\eta_j \in H^2(X(\mathbb{R}),\mathbb{Z})$.*

Proof. The restriction of $i^*(h)$ to the nonorientable part is 2-torsion. We will prove that we can choose a section H transverse to $X(\mathbb{R})$ such that $\#(H\cap M)$ is odd for any nonorientable component. The conclusion will then follow from Lemma 2.2.

As in the proof of lemma 4.11, we will use the surface X^g and the transform T. Let Σ be the finite set of real centers of elementary transformations of T. If M is spherical, $T(M)$ is also spherical. If M is a torus component, then $T(M)$ is a torus component when $\#(\Sigma\cap M)$ is even and a Klein component if $\#(\Sigma\cap M)$ is odd.

Let H' be a section of X^g. The curve $H = T(H')$ is then a section of X. Since Σ is finite, we can move H' to ensure that for all $z \in \rho'(\Sigma)$, $X^g_z(\mathbb{R}) \cap H'(\mathbb{C}) = \varnothing$. Hence the point $p = T(X^g_z)$ is real and belongs to the intersection $H \cap \sigma(H)$.

For each $z \in \rho'(\Sigma)$, the intersection $H\cap\sigma(H)$ is transverse at p and real. Hence H is transverse to $X(\mathbb{R})$ at p. If necessary, we can perturb H to obtain transversality to $X(\mathbb{R})$ at each point.

Now for a non spherical component $T(M)$ of X, the degree of the restriction of $i_!([H])$ to $T(M)$ is equal to the sum of the degree of $i_!([H'])_{|M}$ and $\#(\Sigma\cap M)$.

By Lemma 4.12, $i_!([H'])_{|M} = 0$ for a torus component M of $X^g(\mathbb{R})$ and $\#(\Sigma\cap M)$ is odd when $T(M)$ is a Klein component of $X(\mathbb{R})$. The conclusion follows. $\square$

Given a $\mathbb{R}$-minimal conic bundle X, from Lemmas 4.7 to 4.13 we get

$$H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z}) = \left\langle \bigoplus_{j\in K}\eta_j, \{\eta_j; j\in S\} \right\rangle\ .$$

In other words, the group $H^2_{\mathbb{C}-\mathrm{alg}}(X(\mathbb{R}),\mathbb{Z})$ is generated by the spherical classes and the sum of all the Klein classes. We deduce the theorem:

Theorem 4.14. *Let $X \to B$ be a $\mathbb{R}$-minimal conic bundle. Denote by t the number of torus components of $X(\mathbb{R})$ and by k the number of Klein components. Then*

$$\Gamma(X) = \mathbb{Z}^t \oplus (\mathbb{Z}/2)^{k-1}\ .$$

Corollary 4.15. *Given a real ruling $X \to B$ on a surface with orientable real part $X(\mathbb{R})$, we have $\Gamma(X) = \mathbb{Z}^t$, where t is the number of torus components.*

Proof. The orientability of the real part implies that the real ruling gives rise to a $\mathbb{R}$-minimal conic bundle by making contractions in imaginary fibers only. $\square$

5. Surfaces of negative Kodaira dimension

To prove Theorems 1.1 and 1.2, we will use the following (recall that the subgroup $H^1_{\mathrm{alg}}(V,\mathbb{Z}/2) \subset H^1(V,\mathbb{Z}/2)$ is generated by the cohomology classes Poincaré dual to the homology classes represented by Zariski closed algebraic hypersurfaces of V):

Theorem 5.1 ([Ku99])**.** *Let V be a compact nonsingular real algebraic variety and W be a compact connected nonsingular rational real algebraic surface. Given a $\mathcal{C}^\infty$ map $f\colon V \to W$, the following conditions are equivalent:*

1. *f can be approximated by regular maps;*
2. *f is homotopic to a regular map;*
3. *either W is diffeomorphic to a sphere and*
$$f^*(H^2(W,\mathbb{Z})) \subset H^2_{\mathbb{C}-\mathrm{alg}}(V,\mathbb{Z}/2)\ ,$$
or W is not diffeomorphic to a sphere and
$$f^*(H^1(W,\mathbb{Z})) \subset H^1_{\mathrm{alg}}(V,\mathbb{Z}/2)\ .$$

Given a real algebraic surface X, we denote by t the number of torus components, and by k the number of Klein components. In case X admits a real ruling $\rho\colon X \to B$, we denote by k' the number of Klein components of $X(\mathbb{R})$ whose image by ρ is a connected component of $B(\mathbb{R})$.

Theorem 5.2. *If X is a $\mathbb{C}$-ruled non $\mathbb{C}$-rational real algebraic surface, then*
$$\Gamma(X) = \mathbb{Z}^t \oplus (\mathbb{Z}/2)^{k'-1}\ .$$

Proof. Let Y be a $\mathbb{R}$-minimal model of X, thus the number of Klein components of $Y(\mathbb{R})$ is exactly k' and by Theorem 4.14, $\Gamma(Y) = \mathbb{Z}^t \oplus (\mathbb{Z}/2)^{k'-1}$. The conclusion follows from Proposition 3.5. □

Theorem 5.3. *Let $V = X(\mathbb{R})$ be an orientable real algebraic surface with* $\mathrm{kod}(X) = -\infty$*. Denoting by t the number of components diffeomorphic to a torus, we have*
$$\Gamma(X) = \mathbb{Z}^t\ ,$$
except in case X is the maximal real Del Pezzo surface of degree 2 for which $X(\mathbb{R})$ is the disjoint union of 4 spheres and $\Gamma(X) = \mathbb{Z}/2$.

Remark 5.4. It is an amazing fact that the torus components measure the obstruction to approximate differentiable maps. Indeed, the only case that is known so far is the rational torus $S^1 \times S^1$ realized as the real part of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ endowed with the usual real structure. The proof of $\Gamma(X) = \mathbb{Z}$ uses the torus decomposition as a product of real algebraic curves.

Proof. Since V is orientable, we may assume that X is $\mathbb{R}$-minimal. Thus, we get the conclusion from Theorem 4.15 in case X admits a real ruling and from theorems 1.6 and 1.7 when X is $\mathbb{C}$-rational. □

Corollary 5.5. *Let $V = X(\mathbb{R})$ be an orientable surface of negative Kodaira dimension which is not biregular to a maximal real Del Pezzo surface of degree 2.*

The space of regular maps $\mathcal{R}(V,\mathbb{S}^2)$ is dense in the space of $\mathcal{C}^\infty$ maps $\mathcal{C}^\infty(V,\mathbb{S}^2)$ if and only if all the connected components of V are spherical.

Proof of Theorem 1.1. Theorem 1.1 follows from Theorem 5.1 and Theorem 5.2 □

Proof of Theorem 1.2. We got the conclusion in case W is diffeomorphic to a sphere from Theorem 1.1 and when W is not diffeomorphic to a sphere by Theorem 5.1, and the fact that a connected $\mathbb{C}$-ruled real algebraic surface V satisfies $H^1_{\mathrm{alg}}(V,\mathbb{Z}/2) = H^1(V,\mathbb{Z})$ [Ab00, M03]. □

6. Rational Klein bottles

This short section is devoted to the proof of Theorem 1.3.

Theorem 6.1. *The Klein bottle admits a unique rational model. Namely, the real part of the real Hirzebruch surface $F(1)$.*

Here we can use indifferently the words $\mathbb{C}$-rational or rational because connected $\mathbb{C}$-rational surfaces are $\mathbb{R}$-rational [Si89].

Proof. We want to prove that M is biregularly isomorphic to the real part of the Hirzebruch surface $F(1)$. Let M be a $\mathbb{C}$-rational real algebraic surface diffeomorphic to the Klein bottle. Let X be a $\mathbb{R}$-minimal smooth projective complexification of M. As $M \cong X(\mathbb{R})$ is connected, X is $\mathbb{C}$-minimal and it is a Hirzebruch surface $F(n)$. Furthermore, n is odd and $n > 1$. Indeed, the only $\mathbb{C}$-minimal $\mathbb{R}$-rational surfaces are the real Hirzebruch surfaces $F(n)$ with $n \neq 1$ and $F(n)(\mathbb{R})$ is nonorientable if and only if $n \equiv 1 \mod 2$ [Si89].

Let us denote by H the unique section of the natural real ruling $\rho\colon X \to \mathbb{P}^1$ such that $H^2 = -n$. Choose $\frac{n-1}{2}$ points $p_1, \ldots, p_{\frac{n-1}{2}}$ of H that belong to imaginary fibres of ρ and let $X' = \mathrm{elm}_{p_1,\sigma(p_1)} \circ \cdots \circ \mathrm{elm}_{p_{\frac{n-1}{2}},\sigma(p_{\frac{n-1}{2}})}(X)$.

Then $X'(\mathbb{R})$ is biregularly isomorphic to $X(\mathbb{R})$ and $n' = n - 2(\frac{n-1}{2})$. Furthermore, the transformed surface X' of X is $\mathbb{C}$-isomorphic to $F(1)$. □

References

[Ab00] M. A. Abánades, Algebraic homology for hyperelliptic and real projective ruled surfaces, *Canad. Math. Bull.* **44**, no. 3, 257–265 (2001)

[B78] A. Beauville, *Surfaces algébriques complexes*, Astérisque, vol. 54, Soc. Math. de France, Paris, 1978

[BBK89] J. Bochnak, M. Buchner and W. Kucharz, *Vector bundles over real algebraic varieties*, K-Theory **3**, 271-298 (1989). Erratum, K-Theory **4**, p. 103 (1990)

[BCR98] J. Bochnak, M. Coste, M.-F. Roy, *Real algebraic geometry*, Ergeb. Math. Grenzgeb. (3), vol. 36, Springer-Verlag, 1998

[BK87a] J. Bochnak, W. Kucharz, Algebraic approximation of mappings into spheres, *Michigan Math. J.* **34**, 119–125 (1987)

[BK87b] J. Bochnak, W. Kucharz, Realization of homotopy classes by algebraic mappings, *J. Reine Angew. Math.* **377**, 159–169 (1987)

[Hi64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero.* Ann. of Math. 79, 109-326 (1964)

[Hr76] M. W. Hirsch, *Differential Topology*, Graduate Texts in Math. vol. 33, Springer-Verlag, New York 1976

[HM03] J. Huisman, F. Mangolte, Every orientable Seifert 3-manifold is a real component of a uniruled algebraic variety, *Topology* (to appear)

[JK03] N. Joglar-Prieto, J. Kollár, Real abelian varieties with many line bundles, *Bull. London Math. Soc.* **35**, 79–84 (2003)

[JM03] N. Joglar-Prieto, F. Mangolte, Real algebraic morphisms and Del Pezzo surfaces of degree 2, *Journal of Algebraic Geometry* (to appear)

[Ko97] J. Kollár, Real algebraic surfaces*e-prints, alg–geom/9712003*

[Ku99] W. Kucharz, Algebraic morphisms into rational real algebraic surfaces *J. Algebraic Geometry* **8**, 569–579 (1999)

[M03] F. Mangolte, Cycles algébriques et topologie des surfaces bielliptiques réelles, *Comentarii Mathematici Helvetici* **78**, 385–393 (2003)

[MS74] J. Milnor, J. Stasheff, *Characteristic classes*, Princeton Univ. Press, Princeton 1974

[Si89] R. Silhol, *Real Algebraic Surfaces*, Lecture Notes in Math. **1392**, Springer-Verlag, Berlin 1989

[Sp66] E. H. Spanier, *Algebraic Topology*, Springer-Verlag, New York 1966

Frédéric Mangolte, Laboratoire de Mathématiques, Université de Savoie, 73376 Le Bourget-du-Lac Cedex, France, Tél : (33) 4 79 75 86 60, Fax : (33) 4 79 75 81 42

E-mail address: `mangolte@univ-savoie.fr`